The Maximum Capacity of Folded Origami Containers

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Abstract

We examined which folded origami container has the maximum capacity, defining that a square has the constant area. These containers were compared with a hemisphere, which we set as an ideal container that has the maximum capacity possible. Of all the containers with the same area, we proved that the shuriken model had the largest capacity at, 79.9% of the hemisphere's capacity.

1. Purpose

Our goal was to make a handy container to carry. We decided that it was a container we could fold from a sheet. Therefore, we researched folded square origami containers.

2. Method

We made various kinds of containers with square origami paper and found the maximum capacity of each container by calculating. We defined capacity as the amount of water which we could pour in containers. We did not consider its surface tension. There were three conditions: We could not cut a paper; we could make imaginary containers, it means when we could form it in theory, we saw it as a container even if we could not do it by our hands; and did not consider the thickness of the origami paper. As criteria for comparing the capacitys of each container to find the percentage, we thought of a hemisphere as an ideal. It is well known that a hemisphere has the maximum capacity per constant surface area.

3. Previous research

According to a research conducted by a science club at Kwansei Gakuin Senior High School, they make ogasa figure from paper which had the maximum capacity 81.0% when they could cut the paper[*1].

4. Design of containers

We made these nine containers.
We will state the details later.

(1)Cuboid (2)Tray model (3)Ship model
(4)Hand roll model (5)Column
(6)Pencil model (7)Trigonal Pyramid model
(8)Hozuki model (9)Shuriken model

Nine containers we made

5. Details of the calculation process

v_n is the volume of the nth container and p_n is the volume ratio to the hemisphere.

We used the calculation site WolframAlpha for complicated calculations.

5-0. Hemisphere

This is the hemisphere with a surface area of 1. The radius r equals $\frac{1}{\sqrt{2\pi}}$ from $2\pi r^2 = 1$. Therefore, the volume v_0 is equal to $\frac{1}{3\sqrt{2\pi}}$.

5-1. Cuboid

Figure #1 Cuboid

If the height of the side is set to $x\left(0 < x < \frac{1}{2}\right)$, then

$v_1 = x(1-x)^2$,$v_1{}'$=$(2\,x-1)(6\,x-1)$

When its volume is a maximum, $v_1{}' = 0$

$\therefore\ x = \frac{1}{6}$

$v_1 = \frac{2}{27} \approx 0.07407, p_1 = \frac{2\sqrt{2\pi}}{9} \approx 0.5570$ (55.7%)

5-2. Tray model

Figure #2 Tray model

If the height of the side is set to x and the length of glued parts is set to $\mathrm{t}(0 < \mathrm{t} \le \mathrm{x} < \frac{1}{2})$, then $v_2 = \frac{\sqrt{2xt-t^2}}{6(x-t)}\{(1-2t)^3 - (1-2x)^3\}$

$\frac{d}{dx}v_2 = \frac{t(20t^2+4xt-18x+3)}{3\sqrt{2xt-t^2}}$, $\frac{d}{dt}v_2 = \frac{4x^3+(8t-6)x^2+3(1-2t)^2(x-t)}{3\sqrt{2xt-t^2}}$

Solving the cubed simultaneous equations $\frac{d}{dx}v_2 = \frac{d}{dt}v_2 = 0$, $x \approx 0.2322, \mathrm{t} \approx 0.1088$.

Therefore, $v_2 \approx 0.08642, p_2 \approx 0.6499$ (65.0%).

5-3.Ship model

Figure #3 Ship model

If the length that we show in figure #3 above is set to x(0< x<$\frac{1}{2}$), then

$v_3 = \left(x - \frac{4}{3}x^2\right)\sqrt{\frac{1}{4} - x^2}$, $v_3' = \frac{48x^3-24x^2-8x+3}{6\sqrt{1-4x^2}}$

When v_3' equals 0, $x = \frac{1}{6} - \frac{1}{\sqrt{3}}\cos\frac{4\pi-\alpha}{3}$ $(\alpha \in \mathbb{R}$

$, \sin\alpha = \frac{\sqrt{933}}{36}, \cos\alpha = \frac{11\sqrt{3}}{36}),$ x equals

0.2734 in decimal, so v_3 is equal to 0.7273. $v_3 \approx 0.7273, p_3 \approx 0.5469$ (54.7%)

5-4. Hand roll model

When we pour water in this container, the water surface becomes a circle. Then this structure can be seen as a conus with a generating line's length of 1 and a bottom circumference of $\frac{\pi}{2}$.

Figure #4 Hand roll model

$v_4 = \frac{\sqrt{15}}{192}\pi \approx 0.06337, p_4 = \frac{\sqrt{3\pi^3}}{64} \approx$ 0.4765(47.7%)

5-5. Column

Figure #5 Column

The radius of the bottom's circle is $\frac{1}{2\pi}$ and the height is $\left(1 - \frac{1}{2\pi}\right)$.

$v_5 = \left(\frac{1}{2\pi}\right)^2 \cdot \pi \cdot \left(1 - \frac{1}{2\pi}\right) = \frac{2\pi - 1}{8\pi^2},$

$p_5 = \frac{3\sqrt{2\pi}(2\pi - 1)}{8\pi^2} \approx 0.5032$ (50.3%)

5-6. Pencil model

When we set the length of the generating line of its conus to be x $\left(\frac{1}{2\pi} \leq x \leq 1\right)$,

Figure #6 Pencil model

$v_6 = \frac{1}{4\pi}\left(1 - x + \frac{\sqrt{(2\pi x)^2 - 1}}{6\pi}\right), v_6' = \frac{2\pi x}{3\sqrt{(2\pi x)^2 - a}} - 1$

When its volume becomes maximum, $v_6{}' = 0$

$\therefore\ x = \frac{3\sqrt{2\pi}}{4\pi}$

$v_6 = \frac{1}{4\pi} - \frac{\sqrt{2}}{12\pi^2}, p_6 = \frac{3\sqrt{2\pi}}{4\pi} - \frac{1}{2\sqrt{\pi^3}} \approx 0.5086$ (50.9%)

5-7. Trigonal pyramid model

It's a trigonal pyramid whose bottom is a regular triangle and the height is $\frac{1}{2}$.

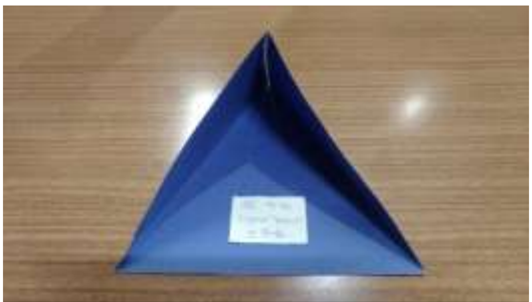


Figure #7 Trigonal pyramid model

$v_7 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{24}, p_7 = \frac{\sqrt{6\pi}}{8} \approx 0.5427$(54.3%)

5-8. Hozuki model

Figure #8 Hozuki model

This is a container in which the diagonal section of the origami is inscribed in the hemisphere of radius $\frac{\sqrt{2}}{\pi}$ and the container cross section becomes square when cut parallel to the section of the hemisphere.

The square area of the mouth of the container is $\left(\frac{\sqrt{2}}{\pi}\cdot\sqrt{2}\right)^2=\frac{4}{\pi^2}$. The area of the circle of radius $\frac{\sqrt{2}}{\pi}$ is $\frac{2}{\pi}$, so the area ratio between the cut and the circle is $\frac{4}{\pi^2}:\frac{2}{\pi}=\frac{2}{\pi}:1$.

However, it is not possible to flatten the mouth of the container. Therefore, we makes the formula of the volume of the container by setting the height of the lowest point as h.

$$\frac{2}{\pi}\cdot\int_{\frac{\sqrt{2}}{\pi}-h}^{\frac{\sqrt{2}}{\pi}}\pi\left(\sqrt{\left(\frac{\sqrt{2}}{\pi}\right)^2-x^2}\right)^2 dx=\frac{2\sqrt{2}}{\pi}h^2-\frac{2}{3}h^3$$

We found the height h by using Excel.

Expressing an ellipse with a short radius $\frac{1}{\pi}$ and a long radius $\frac{\sqrt{2}}{\pi}$ in polar coordinates,

$$r(\theta)=\frac{1}{\pi\sqrt{\frac{\cos^2\theta}{2}+\sin^2\theta}}$$

Now, assuming that 2π is divided into d, the angle per unit is $\frac{2\pi}{d}$. The distance between $r(\theta)$ and $r\left(\theta+\frac{2\pi}{d}\right)$ is

$$\sqrt{\left(r\left(\theta+\frac{2\pi}{d}\right)\sin\frac{2\pi}{d}\right)^2+\left(r(\theta)-r\left(\theta+\frac{2\pi}{d}\right)\cos\frac{2\pi}{d}\right)^2}=\sqrt{r(\theta)^2-2r(\theta)r\left(\theta+\frac{2\pi}{d}\right)\cos\frac{2\pi}{d}+r\left(\theta+\frac{2\pi}{d}\right)^2}$$

When we add $\frac{2\pi}{d}$ $(n-1)$ times from $\theta=0$, the approximate length L is

$$\sum_{k=0}^{n}\sqrt{r\left(\frac{2\pi n}{d}\right)^2-2r\left(\frac{2\pi n}{d}\right)r\left(\frac{2\pi(n+1)}{d}\right)\cos\frac{2\pi}{d}+r\left(\frac{2\pi(n+1)}{d}\right)^2}$$

We find the maximum natural number $n(0<n\le d-1)$ where $\mathrm{L}\ge 1$. At this time, the height of the container is $\frac{\sqrt{2}}{\pi}-r\left(\frac{2\pi n}{d}\right)\cos\frac{2\pi n}{d}$.

When the height and volume ratio were calculated by changing the number of divisions, the results are shown in Table on the right. From this, it was found that the volume ratio of the ground lamp type was 59.3%.

d	height	volume	ratio
16	0.45016	0.12163	0.91463
32	0.38746	0.09638	0.72478
64	0.35575	0.08393	0.63111
128	0.35575	0.08393	0.63111
256	0.34776	0.08084	0.60793
512	0.34376	0.07931	0.59639
1024	0.34376	0.07931	0.59639
2048	0.34275	0.07892	0.59350
4096	0.34275	0.07892	0.59350
8192	0.34275	0.07892	0.59350
16384	0.34275	0.07892	0.59350
32767	0.34270	0.07891	0.59336

The relation between d and the volume ratio

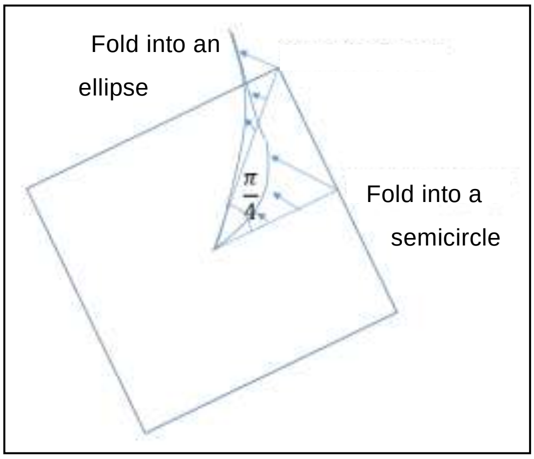
5-9. Shuriken model

This is a container in which the part connecting the middle points of the opposite sides of the origami is inscribed in a semicircle with a radius of $\frac{1}{\pi}$, The other part of the side is lifted so that the mouth of the container is flat.

Since the container is a point-symmetrical figure, we find the equation for the major radius of the ellipse, which is the cross section when we cut a plane containing the axis of symmetry and find the volume by diriving an equation of the mouth of the container.

At polar coordinates $(\theta, r(\theta))$ where the center of gravity of the origami is the origin and the axis and one side crossed vertically, the part of $\theta = 0$ folds into a semicircle and the part of $0 < \theta \leq \frac{\pi}{4}$ folds into an ellipse. The other parts also fold in the same manner as $0 < \theta \leq \frac{\pi}{4}$.

Figure #9 Shuriken model



We used three approximations[*2] where C is the circumference of an ellipse with a long radius a and a short radius b.

$$C = 2\pi a\left(1 - \frac{1}{4}\left(1 - \frac{b^2}{a^2}\right)\right) \quad \cdots ①$$

$$C = \pi(a+b)\left(1 + \frac{1}{4}\left(\frac{a-b}{a+b}\right)^2\right) \quad \cdots ②$$

$$C = \pi\left(3(a+b) - \sqrt{(3a+b)(a+3b)}\right) \quad \cdots ③$$

Simplifying these for a,

$$①a = \frac{C}{3\pi}\left(1 + \sqrt{1 - 3\left(\frac{b}{C}\pi\right)^2}\right)$$

$$②a = \frac{-3\pi b + 2C + 2\sqrt{-4\pi^2 b^2 + 2\pi bC + C^2}}{5\pi}$$

$$③a = \frac{-4\pi b + 3C + \sqrt{-20\pi^2 b^2 + 12\pi bC + 3C^2}}{6\pi}$$

Substituting $b = \frac{1}{\pi}$ and $C = \frac{2}{\cos\theta}$ into these equations,

$$①a = \frac{2 + \sqrt{3 - \cos^2\theta}}{3\pi\cos\theta}$$

$$②a = \frac{4 - 3\cos\theta + 2\sqrt{\cos\theta + \sin^2\theta}}{5\pi\cos\theta}$$

$$③a=\frac{3-2\cos\theta+\sqrt{3+6\cos\theta-5\cos^2\theta}}{6\pi\cos\theta}$$

Substituting $\theta=\frac{\pi}{4}$ into these equations,

$$①a=\frac{2\sqrt{2}+\sqrt{5}}{3\pi}\approx 0.537360$$

$$②a=\frac{4\sqrt{2}-3+4\sqrt{1+\sqrt{2}}}{5\pi}\approx 0.564806$$

$$③a=\frac{3\sqrt{2}-2+\sqrt{1+6\sqrt{2}}}{3\pi}\approx 0.564730$$

According to the calculation using a computer (Newton-Raphson method), the major radius of an ellipse with a short radius of $\frac{1}{\pi}$ and a perimeter of $2\sqrt{2}$ is 0.5647.

Therefore, in the calculation of the volume, we use the equation ③ which has a similar result.

$$a=\frac{3-2\cos\theta+\sqrt{3+6\cos\theta-5\cos^2\theta}}{6\pi\cos\theta}$$

The area of the mouth is $4\int_0^{\frac{\pi}{4}} a^2\,d\theta\approx 0.500474$.

The ratio between the area of the mouth and that of the circumscribed circle is

$$0.500474/\left\{\pi\left(\frac{3\sqrt{2}-2+\sqrt{1+6\sqrt{2}}}{3\pi}\right)^2\right\}=\alpha$$

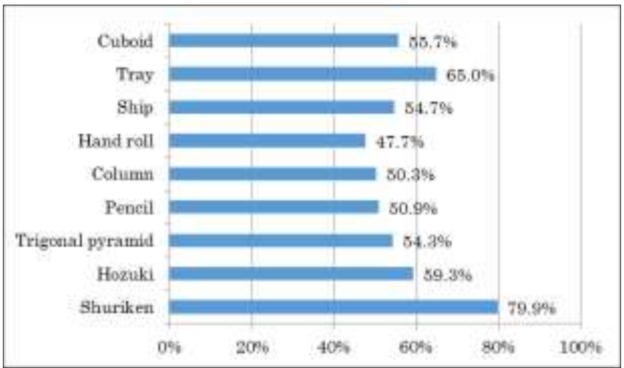
The volume of the ellipsoid circumscribing the container is $\frac{2}{3}\pi\cdot\frac{1}{\pi}\left(\frac{3\sqrt{2}-2+\sqrt{1+6\sqrt{2}}}{3\pi}\right)^2=\beta$

The volume ratio of the shuriken model was found to be 79.9%.

$v_9=\alpha\beta\approx 0.1062, p_9=3\sqrt{2\pi}\approx 0.7986$ (79.9%)

6. Conclusion and consideration

The volume ratio is illuminated in the graph on the right. In our research, the shuriken model has the maximum volume ratio, the tray model has the second, and the hozuki model has the third.



The volume ratio comparing with hemisphere

The reasons why it shows this conclusion are explained with two ideas. One is the idea that those models are rounder than the others. The other is that those containers have less useless part, which is nothing to do with the volume.

Namely, it is expected that the closer a structure of a container become to hemisphere, the more capacious it become. Therefore, the tray model has larger capacity than the cuboid, and the shuriken model has larger capacity than the tray model. However, the capacity of the hozuki model is smaller than that of the tray model. This is because the hozuki model has some useless parts in itself to contribute to increasing capacity.

７． Assignment in future

These are two things which we have to do from now on.

Firstly, we have to establish a method which proves a container has the maximum capacity of all existing containers. We need to study from another angle because we cannot prove it with our method.

Secondly, there is a difference between the shuriken model which we assume in theory and an actual handmade one. Although we expected that the length of the diagonal of it would be 0.565, in fact, it is 0.61. Therefore, the volume will be larger but we cannot prove it, so we need to study this in the future.

８． References

[*1] 小笠図形の体積:数−学(http://ma-3517hm.cocolog-nifty.com/blog/2013/07/post-ef55.html)

[*2] Wikipedia-楕円(https://ja.wikipedia.org/wiki/楕円)

９． Key words

Origami　Maximum capacity　Shuriken model